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A UNIFIED TREATMENT OF LUNAR THEORY AND ARTIFICIAL SATELLITE THEORY

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16. Abstract

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A UNIFIED TREATMENT OF LUNAR THEORY AND ARTIFICIAL SATELLITE THEORY

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SUMMARY

Lunar theory and artificial satellite theory are treated by a unified method that is a generalization of the von Zeipel procedure. The technique of separation of variables is used to generate a single canonical transformation that eliminates the time and all the angle variables from the Hamiltonian. The validity of the method is established and the calculations are carried far enough to illustrate the techniques involved.

INTRODUCTION

The theory of artificial satellite motion has been developed in recent years by methods that had not previously been considered applicable to lunar theory. The problems appear quite different, mathematically, because of the explicit appearance of the time in the lunar problem.

In the present paper the familiar von Zeipel procedure is generalized in two respects. Explicit dependence on the time is permitted, in a certain restricted form, and then the time and all the angle variables are removed from the Hamiltonian by means of a single canonical transformation. This generalization is capable of solving the complete lunar problem far more efficiently than the older methods of von Zeipel or Delaunay. It is then shown that this method can be applied directly, in a simplified, degenerate form, to the artificial satellite problem.

The familiar Delaunay variables are used, with one modification. The longitude of the node is measured from a moving, rather than from an inertial reference direction. For lunar theory the reference direction is that of the sun, for artificial satellite theory, it is the meridian of Greenwich. This device removes the explicit time dependence from the satellite problem, and drastically simplifies the lunar problem.

The generalized procedure is presented in the next section, and its application to lunar and satellite theory follow in that order.

SYMBOLS

a	semimajor axis
e	eccentricity
f	true anomaly
g	argument of perigee
h	longitude of node from moving reference line
k	gaussian gravitational constant
l	mean anomaly
m	dimensionless ratio of mean motions
n	mean motion of moon or satellite
p	angle variable
q	action variable
r	geocentric distance
t	time
u	eccentric anomaly
x	action variable
y	angle variable
B	earth-moon barycenter
$C_{n,m}$, $S_{n,m}$	dimensionless coefficients in tesseral harmonics
D_n	zonal harmonic
E	new Hamiltonian, mass of earth
F	old Hamiltonian
G	$\sqrt{\mu a(1 - e^2)}$
H	$G \cos I$
I	inclination angle
J_n	dimensionless coefficients in zonal harmonics
L	$\sqrt{\mu a}$
M	mass of moon
P_n	Legendre polynomial
P_n^m	associated Legendre function
R	equatorial radius of the earth
S	determining function, mass of sun
$T_{n,m}$	tesseral harmonic
W	determining function
X,Y,Z	decomposition types
β, δ	solar factors
γ	$1 - \cos I$
Γ	longitude of sun's perigee
ε	eccentricity of sun's orbit
θ	geocentric latitude
λ	mean anomaly of sun
Λ	geographic longitude
μ	gravitational parameter
ν	mean motion of sun, spin velocity of earth

ρ distance from sun to earth-moon barycenter

ϕ true anomaly of sun

ψ elongation of moon

Ω longitude of node from inertial reference line

THE GENERALIZED VON ZEIPEL TRANSFORMATION

Recent applications of the von Zeipel transformation have been restricted essentially to Hamiltonians in which the time does not occur explicitly (refs. 1 and 2). When it does occur, it is formally removed by the artifice of adjoining an additional pair of canonical variables (ref. 3, pp. 530-531, and ref. 4). A generalized method is developed here that is applicable to certain problems involving a time-dependent Hamiltonian. In the case of lunar theory this permits many of the solar terms to be expressed in closed form, thereby avoiding certain Fourier series expansions that frequently complicate the problem unnecessarily.

Consider the canonical system

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}$$

with "action variables" $x = (x_1, x_2, x_3)$ and "angle variables" $y = (y_1, y_2, y_3)$. Let the Hamiltonian be of the form

$$F = F(x, y, \lambda)$$

where λ (an angle) is a linear function of the time, with $\nu = d\lambda/dt$ a constant. This is the only form of time-dependence that will be considered.

Now let $q = (q_1, q_2, q_3)$ and $p = (p_1, p_2, p_3)$ be new canonical variables defined by the implicit transformation equations

$$p_i = \frac{\partial W(q, y, \lambda)}{\partial q_i}$$

$$x_i = \frac{\partial W(q, y, \lambda)}{\partial y_i}$$

with the "determining function," W, given by

$$W = q_i y_i + S(q, y, \lambda)$$

(Einstein's summation convention will be used throughout this paper: a repeated subscript is to be summed over its range.) Thus, in vector notation,

$$x = q + \Delta q$$

$$y = p + \Delta p$$

with the increments having components given implicitly by

$$(\Delta q)_i = \frac{\partial S(q, y, \lambda)}{\partial y_i}$$

$$(\Delta p)_i = -\frac{\partial S(q, y, \lambda)}{\partial q_i}$$

Expanding the right member of the last equation in a Taylor series near $y = p$ gives

$$(\Delta p)_i = -\frac{\partial S(q, p, \lambda)}{\partial q_i} - \frac{\partial^2 S}{\partial q_i\ \partial p_j} (\Delta p)_j - \frac{1}{2} \frac{\partial^3 S}{\partial q_i\ \partial p_j\ \partial p_k} (\Delta p)_j (\Delta p)_k \ . \ . \ .$$

This can be solved iteratively to any desired degree of accuracy, if S and its derivatives are assumed small:

First order: $$(\Delta p)_i = -\frac{\partial S}{\partial q_i}$$

Second order: $$(\Delta p)_i = -\frac{\partial S}{\partial q_i} - \frac{\partial^2 S}{\partial q_i\ \partial p_j}\left(-\frac{\partial S}{\partial q_j}\right)$$

Third order: $$(\Delta p)_i = -\frac{\partial S}{\partial q_i} - \frac{\partial^2 S}{\partial q_i\ \partial p_j}\left(-\frac{\partial S}{\partial q_j} + \frac{\partial^2 S}{\partial q_j\ \partial p_k}\frac{\partial S}{\partial q_k}\right)$$

$$- \frac{1}{2}\frac{\partial^3 S}{\partial q_i\ \partial p_j\ \partial p_k}\left(-\frac{\partial S}{\partial q_j}\right)\left(-\frac{\partial S}{\partial q_k}\right)$$

Similarly,

$$(\Delta q)_i = \frac{\partial S}{\partial p_i} - \frac{\partial^2 S}{\partial p_i\ \partial p_j}\frac{\partial S}{\partial q_j} + \frac{\partial^2 S}{\partial p_i\ \partial p_j}\frac{\partial^2 S}{\partial q_j\ \partial p_k}\frac{\partial S}{\partial q_k}$$

$$+ \frac{1}{2}\frac{\partial^3 S}{\partial p_i\ \partial p_j\ \partial p_k}\frac{\partial S}{\partial q_j}\frac{\partial S}{\partial q_k} \ . \ . \ .$$

Thus the equations

$$x = q + \Delta q$$

$$y = p + \Delta p$$

with Δq, Δp given above, constitute an explicit solution of the implicit transformation equations.

Now, q and p are canonical variables:

$$\frac{dq_i}{dt} = \frac{\partial E}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial E}{\partial q_i}$$

with the new Hamiltonian, E, given implicitly by

$$E = F(x, y, \lambda) - \frac{\partial W(q, y, \lambda)}{\partial t} = F - \nu \frac{\partial S(q, y, \lambda)}{\partial \lambda}$$

(ref. 5, ch. 6)

If the right member is again expanded in a Taylor series and the explicit forms of Δq and Δp are inserted, an explicit representation $E(q, p, \lambda)$ is obtained. Successive differentiations with respect to p can be used to yield the symmetric equation

$$E(q, p, \lambda) + \frac{\partial E}{\partial p_i}\frac{\partial S}{\partial q_i} + \frac{1}{2}\frac{\partial^2 E}{\partial p_i\,\partial p_j}\frac{\partial S}{\partial q_i}\frac{\partial S}{\partial q_j} + \frac{1}{6}\frac{\partial^3 E}{\partial p_i\,\partial p_j\,\partial p_k}\frac{\partial S}{\partial q_i}\frac{\partial S}{\partial q_j}\frac{\partial S}{\partial q_k} + \ldots$$

$$= F(q, p, \lambda) - \nu\frac{\partial S}{\partial \lambda} + \frac{\partial F}{\partial q_i}\frac{\partial S}{\partial p_i} + \frac{1}{2}\frac{\partial^2 F}{\partial q_i\,\partial q_j}\frac{\partial S}{\partial p_i}\frac{\partial S}{\partial p_j}$$

$$+ \frac{1}{6}\frac{\partial^3 F}{\partial q_i\,\partial q_j\,\partial q_k}\frac{\partial S}{\partial p_i}\frac{\partial S}{\partial p_j}\frac{\partial S}{\partial p_k} + \ldots$$

and only the new variables q and p appear.

Now let n be the mean motion of the body being studied (moon or satellite in the present paper) and let $m = \nu/n$ be a small quantity (i.e., $m \ll 1$). Introducing series expansions

$$F = \sum_{k=o}^{\infty} F_k \qquad E = \sum_{k=o}^{\infty} E_k \qquad S = \sum_{k=1}^{\infty} S_k$$

$$\Delta q = \sum_{k=1}^{\infty} \Delta_k q \qquad \Delta p = \sum_{k=1}^{\infty} \Delta_k p$$

with the subscript, k, denoting $O(m^k)$, permits the separation of the equations according to order of magnitude, that is, according to powers of m.

Thus, the explicit transformation equations become

$$(\Delta_1 p)_i = - \frac{\partial S_1}{\partial q_i}$$

$$(\Delta_2 p)_i = - \frac{\partial S_2}{\partial q_i} + \frac{\partial^2 S_1}{\partial q_i \; \partial p_j} \frac{\partial S_1}{\partial q_j}$$

$$\begin{aligned}(\Delta_3 p)_i = &- \frac{\partial S_3}{\partial q_i} + \frac{\partial^2 S_2}{\partial q_i \; \partial p_j} \frac{\partial S_1}{\partial q_j} + \frac{\partial^2 S_1}{\partial q_i \; \partial p_j} \frac{\partial S_2}{\partial q_j} \\ &- \frac{\partial^2 S_1}{\partial q_i \; \partial p_j} \frac{\partial^2 S_1}{\partial q_j \; \partial p_k} \frac{\partial S_1}{\partial q_k} \\ &- \frac{1}{2} \frac{\partial^3 S_1}{\partial q_i \; \partial p_j \; \partial p_k} \frac{\partial S_1}{\partial q_j} \frac{\partial S_1}{\partial q_k}\end{aligned}$$

and so forth, for Δp, with analogous equations for Δq.

The equation for the Hamiltonian separates into the "von Zeipel equations":

$$E_o = F_o$$

$$E_1 + \frac{\partial E_o}{\partial p_i} \frac{\partial S_1}{\partial q_i} = F_1 + \frac{\partial F_o}{\partial q_i} \frac{\partial S_1}{\partial p_i}$$

$$E_2 + \frac{\partial E_o}{\partial p_i} \frac{\partial S_2}{\partial q_i} + \frac{\partial E_1}{\partial p_i} \frac{\partial S_1}{\partial q_i} + \frac{1}{2} \frac{\partial^2 E_o}{\partial p_i \; \partial p_j} \frac{\partial S_1}{\partial q_i} \frac{\partial S_1}{\partial q_j}$$

$$= F_2 - \nu \frac{\partial S_1}{\partial \lambda} + \frac{\partial F_o}{\partial q_i} \frac{\partial S_2}{\partial p_i} + \frac{\partial F_1}{\partial q_i} \frac{\partial S_1}{\partial p_i} + \frac{1}{2} \frac{\partial^2 F_o}{\partial q_i \; \partial q_j} \frac{\partial S_1}{\partial p_i} \frac{\partial S_1}{\partial p_j}$$

$$E_3 + \frac{\partial E_o}{\partial p_i}\frac{\partial S_3}{\partial q_i} + \frac{\partial E_1}{\partial p_i}\frac{\partial S_2}{\partial q_i} + \frac{\partial E_2}{\partial p_i}\frac{\partial S_1}{\partial q_i} + \frac{\partial^2 E_o}{\partial p_i\,\partial p_j}\frac{\partial S_1}{\partial q_i}\frac{\partial S_2}{\partial q_j} + \frac{1}{2}\frac{\partial^2 E_1}{\partial p_i\,\partial p_j}\frac{\partial S_1}{\partial q_i}\frac{\partial S_1}{\partial q_j}$$

$$+ \frac{1}{6}\frac{\partial^3 E_o}{\partial p_i\,\partial p_j\,\partial p_k}\frac{\partial S_1}{\partial q_i}\frac{\partial S_1}{\partial q_j}\frac{\partial S_1}{\partial q_k}$$

$$= F_3 - \nu\frac{\partial S_2}{\partial \lambda} + \frac{\partial F_o}{\partial q_i}\frac{\partial S_3}{\partial p_i} + \frac{\partial F_1}{\partial q_i}\frac{\partial S_2}{\partial p_i} + \frac{\partial F_2}{\partial q_i}\frac{\partial S_1}{\partial p_i} + \frac{\partial^2 F_o}{\partial q_i\,\partial q_j}\frac{\partial S_1}{\partial p_i}\frac{\partial S_2}{\partial p_j}$$

$$+ \frac{1}{2}\frac{\partial^2 F_1}{\partial q_i\,\partial q_j}\frac{\partial S_1}{\partial p_i}\frac{\partial S_1}{\partial p_j} + \frac{1}{6}\frac{\partial^3 F_o}{\partial q_i\,\partial q_j\,\partial q_k}\frac{\partial S_1}{\partial p_i}\frac{\partial S_1}{\partial p_j}\frac{\partial S_1}{\partial p_k}$$

and so forth.

It may be remarked that this representation of the transformation equations and of the Hamiltonian explicitly in terms of the new variables q and p is not the usual practice. Brouwer (ref. 1) uses the mixed set of variables q and y.

In references 1, 2, and 4 the von Zeipel transformation is used successively to eliminate only one angle variable, p_i, at a time. In the applications in the present paper, a single transformation will be exhibited that eliminates λ and all three angle variables simultaneously. Thus, the partial derivatives disappear from the left members of the von Zeipel equations, giving simply the components, E_k, explicitly. The equations will be used here only in this simplified form.

These equations can be regarded as a simultaneous set of partial differential equations for the components, S_k, of the determining function, S. A technique that is similar to the classical one of separation of variables will be used to construct a recursive set of equations, each of which is a linear, first-order partial differential equation with constant coefficients. Specifically, the angle variables, p, will be the Delaunay variables (l, g, h), respectively, the mean anomaly, argument of perigee, and longitude of the node. Each term, S_k, will be decomposed into the sum of three terms

$$S_k = X_k + Y_k + Z_k$$

where

X a periodic function of l

Y a periodic function of g, h, and λ, with l absent and the sum, $h + \lambda$, excluded

Z a periodic function of two variables, g and the sum $h + \lambda$, with l absent

Similarly, each term, F_k, of the Hamiltonian will be decomposed into the sum of four terms.

$$F_k = XF_k + YF_k + ZF_k + EF_k$$

with the prefix E denoting a function that is independent of λ and all the angle variables (l, g, h), the other prefixes having the meaning assigned above. The calculations will then proceed according to the scheme:

Order 0:	E_o			
Order 1:	E_1	X_1		
Order 2:	E_2	X_2	Y_1	
Order $k \geq 3$:	E_k	X_k	Y_{k-1}	Z_{k-2}

This is reminiscent of the schemes of Brouwer and Hori (refs. 1 and 2) which yield short-period terms to a higher order than long-period terms. Later it will be shown that the components X, Y, Z are obtained as follows:

The component X_k will be obtained by simple quadrature:

$$X_k = \int (\text{function of type } X) dl$$

The component Y_k will satisfy an equation of the type

$$\frac{\partial Y}{\partial \lambda} - \frac{\partial Y}{\partial h} = f(\theta)$$

$$\theta = i\lambda + jh + kg, \qquad i \neq j, \; i, j, k \text{ integers}$$

Clearly, the solution is

$$Y = \frac{1}{i - j} \int f(\theta) d\theta$$

The component Z_k will satisfy an equation of the form

$$\xi \frac{\partial Z}{\partial g} + \eta \frac{\partial Z}{\partial h} = f(\theta)$$

$$\theta = ig + j(h + \lambda), \qquad i, j \text{ integers}$$

and the solution is

$$Z = \frac{1}{\xi i + \eta j} \int f(\theta) d\theta$$

where ξ and η are functions of the action variables (L, G, H) and it will be proved that the denominator never vanishes, so that no critical case occurs in the lunar theory. In satellite theory, of course, the case of critical inclination has to be excluded.

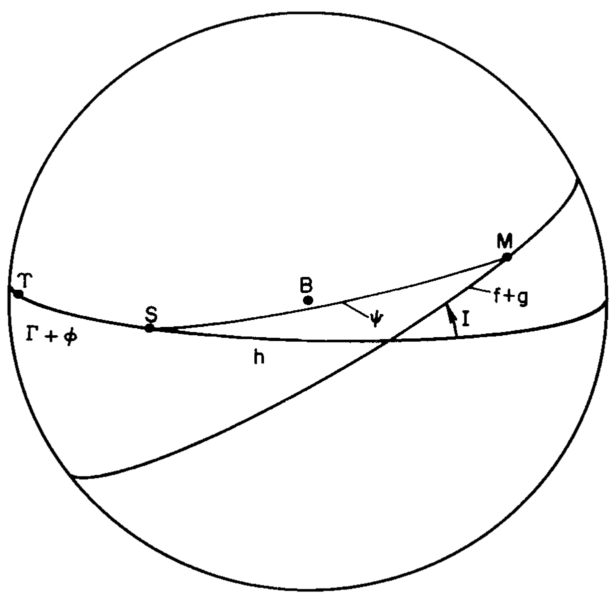
Lunar theory will be treated first, and then it will be shown that satellite theory is simply a degenerate case.

It is clear from the preceding discussion that the first task is to obtain a series representation of the Hamiltonian, in terms of the Delaunay variables, and then to exhibit the decomposition by type (X, Y, Z, E). This will be accomplished in the next section.

LUNAR THEORY

Development of the Hamiltonian

The lunar theory to be developed here is that of the motion of the moon about the earth, under the assumption that the earth-moon barycenter (B) moves about the sun in a Keplerian orbit, the plane of which will be referred to as the ecliptic. Earth, moon and sun are regarded as point masses, and planetary perturbations are ignored. The Hamiltonian for the classical Delaunay variables (L, G, H) and (l, g, Ω) is



$$F' = \frac{\mu^2}{2L^2} + \frac{k^2 S}{\rho^3} r^2 P_2(\cos \psi)$$

$$+ \frac{k^2 S}{\rho^4} \frac{E - M}{E + M} r^3 P_3(\cos \psi) \; . \; . \; .$$

(see ref. 3, pp. 271 and 291), and the law of cosines of spherical trigonometry gives (see sketch):

$$\cos \psi = \cos(f + g)\cos h$$

$$- \sin(f + g)\sin h \cos I$$

where h is the elongation of the node, measured from the sun:

$$h = \Omega - (\Gamma + \phi)$$

If h is used, rather than Ω, as the conjugate variable to H, then the Hamiltonian is simply

$$F = F' + H \frac{d\phi}{dt}$$

since

$$\frac{dh}{dt} = \frac{d\Omega}{dt} - \frac{d\phi}{dt} = - \frac{\partial F'}{\partial H} - \frac{\partial}{\partial H} \left(\frac{d\phi}{dt} \cdot H \right) = - \frac{\partial F}{\partial H}$$

and

$$\frac{dH}{dt} = \frac{\partial F'}{\partial \Omega} = \frac{\partial F'}{\partial h} = \frac{\partial F}{\partial h}$$

This choice of variables makes ψ independent of the time, so that solar effects enter only via ρ, the distance from the sun to the earth-moon barycenter. Many Fourier series expansions and multiplications are thus avoided.

The usual methods of the theory of elliptic motion (ref. 3, ch. 2, and ref. 6, ch. 3) can be used to obtain the solar factors in closed form and in series. The only ones needed here are

$$\frac{d\phi}{dt} = \nu(1 + \delta)$$

$$\frac{k^2 S}{\rho^3} = \nu_2(1 + \beta)$$

where ν is the sun's mean motion and

$$\nu_2 = \nu\nu_1, \qquad \nu_1 = \frac{S}{S + E + M} \cdot \frac{\nu}{(1 - \varepsilon^2)^{3/2}}$$

$$\delta = \frac{d(\phi - \lambda)}{d\lambda} = \sum_{k=1}^{\infty} \delta_k \cos k\lambda$$

$$\beta = \frac{d(\phi - \lambda + \varepsilon \sin \phi)}{d\lambda} = \sum_{k=1}^{\infty} \beta_k \cos k\lambda$$

and δ_k, β_k are power series starting with ε^k; for example,

$$\delta_1 = 2\varepsilon - \frac{1}{4}\varepsilon^3 \ . \ . \ .$$

$$\delta_2 = \frac{5}{2}\varepsilon^2 - \frac{11}{24}\varepsilon^4 \ . \ . \ .$$

$$\beta_1 = 3\varepsilon - \frac{9}{8}\varepsilon^3 \ . \ . \ .$$

$$\beta_2 = \frac{9}{2}\varepsilon^2 - \frac{13}{4}\varepsilon^4 \ . \ . \ .$$

To obtain the desired form for the lunar factors, write

$$\cos\psi = \cos(f + g + h) + \gamma \sin(f + g)\sin h$$

where

$$\gamma = 1 - \cos I = 1 - \frac{H}{G}$$

Then the Legendre polynomials can be written as

$$P_n(\cos\psi) = \sum_{k=o}^{n} p_{n,k}\gamma^k$$

where $p_{n,k}$ is a trigonometric polynomial in f, with coefficients that are trigonometric functions of g and h; for example,

$$p_{2,o} = \frac{1}{4} + \frac{3}{4}\cos(2g + 2h)\cos 2f - \frac{3}{4}\sin(2g + 2h)\sin 2f$$

$$p_{2,1} = -\frac{3}{2}\sin^2 h + \frac{3}{2}\sin h \sin(2g + h)\cos 2f + \frac{3}{2}\sin h \cos(2g + h)\sin 2f$$

$$p_{2,2} = \frac{3}{4}\sin^2 h - \frac{3}{4}\sin^2 h \cos 2g \cos 2f + \frac{3}{4}\sin^2 h \sin 2g \sin 2f$$

The Hamiltonian can now be written as a series

$$F = \sum_{k=o}^{\infty} F_k$$

where

$$F_o = \frac{\mu^2}{2L^2}$$

$$F_1 = \nu H$$

$$F_2 = \nu\delta H + \nu_2 r^2 p_{2,o}$$

$$F_3 = \nu_2 \beta r^2 p_{2,o} + \nu_2 \gamma r^2 p_{2,1}$$

and so forth, under the assumptions that $\varepsilon = 0(m)$, $\gamma = 0(m)$.

Finally, to separate each term by type (X, Y, Z, E) note that the true anomaly, f, enters in the form

$$r^n \cos kf,\ r^n \sin kf, \qquad 0 \leq k \leq n$$

and these can be expressed as trigonometric polynomials in u, the eccentric anomaly, the coefficients being functions of a and e (see refs. 2 and 4). For example,

$$r^2 = a^2\left(1 + \frac{3}{2} e^2\right) + A_o$$

$$r^2 \cos 2f = \frac{5}{2} a^2e^2 + B_o$$

$$r^2 \sin 2f = C_o$$

where

$$A_o = a^2\left[-2e\left(\frac{1}{2} e + \cos u\right) + \frac{1}{2} e^2 \cos 2u\right]$$

$$B_o = a^2\left[-2e\left(\frac{1}{2} e + \cos u\right) + \left(1 - \frac{1}{2} e^2\right)\cos 2u\right]$$

$$C_o = a^2\sqrt{1 - e^2}\ (-2e \sin u + \sin 2u)$$

and each of these is periodic in l, the mean anomaly, with vanishing mean value (type X).

The desired representation of the Hamiltonian, then, is

$$F_o = \frac{\mu^2}{2L^2} \qquad \text{Type E}$$

$$F_1 = \nu H \qquad \text{Type E}$$

F_2:

$$XF_2 = \frac{1}{4} \nu_2\left[A_o + 3B_o \cos(2g + 2h) - 3C_o \sin(2g + 2h)\right]$$

$$YF_2 = \nu\delta H + \frac{15}{8} \nu_2 a^2e^2 \cos(2g + 2h)$$

$$ZF_2 = 0$$

$$EF_2 = \frac{1}{4} \nu_2 a^2\left(1 + \frac{3}{2} e^2\right)$$

$$F_3: \quad XF_3 = \beta \cdot XF_2 + \frac{3}{2} \gamma\nu_2 \sin h[-A_o \sin h + B_o \sin(2g + h) + C_o \cos(2g + h)]$$

$$YF_3 = \frac{15}{16} \nu_2\beta_1 a^2e^2[\cos(2g + 2h + \lambda) + \cos(2g + 2h - \lambda)]$$

$$+ \frac{3}{4} \gamma\nu_2 a^2 \left[\left(1 + \frac{3}{2} e^2\right)\cos 2h - \frac{5}{2} e^2 \cos(2g + 2h)\right] + \beta \cdot EF_2$$

$$ZF_3 = \frac{15}{8} \gamma\nu_2 a^2e^2 \cos 2g$$

$$EF_3 = - \frac{3}{4} \gamma\nu_2 a^2\left(1 + \frac{3}{2} e^2\right)$$

$$F_4: \quad ZF_4 = \frac{15}{16} \nu_2\beta_2 a^2e^2 \cos(2g + 2h + 2\lambda) - \frac{15}{16} \gamma^2\nu_2 a^2e^2 \cos 2g$$

$$EF_4 = \frac{3}{8} \gamma^2\nu_2 a^2\left(1 + \frac{3}{2} e^2\right)$$

The X and Y components of F_4 will not be given, since the analysis will not be carried far enough in the present paper to require them.

It may be remarked that F_3 contains the term $\nu_2\beta r^2 p_{2,o}$, which yields, among others, the term

$$\frac{15}{8} \nu_2\beta a^2e^2 \cos(2g + 2h)$$

and here the series for β must be used, giving

$$\frac{15}{16} \nu_2 a^2e^2\beta_1[\cos(2g + 2h + \lambda) + \cos(2g + 2h - \lambda)]$$

$$+ \frac{15}{16} \nu_2 a^2e^2\beta_2[\cos(2g + 2h + 2\lambda) + \cos(2g + 2h - 2\lambda)]$$

$$+ \ . \ . \ .$$

The first line has been included in YF_3, and the first term of the second line is included in ZF_4. There are also higher order contributions that must be properly assigned in a complete theory.

Solution of the Lunar Problem

The lunar problem can now be solved by means of the generalized von Zeipel transformation from the old variables

$$x = (L, G, H), \qquad y = (l, g, h)$$

to the new variables

$$q = (\overline{L}, \overline{G}, \overline{H}), \qquad p = (\overline{l}, \overline{g}, \overline{h})$$

where the bar notation is used to emphasize the fact that the new variables are the mean values of the old ones, in the usual astronomical sense. That is, the difference between each new variable and the corresponding old one is a sum of terms each of which is a periodic function of the time with vanishing mean value.

Since the new Hamiltonian contains only the action variables q, the new canonical equations are

$$\frac{dq_i}{dt} = \frac{\partial E}{\partial p_i} = 0$$

so that the q's $(\overline{L}, \overline{G}, \overline{H})$ are constants. Then

$$\frac{dp_i}{dt} = - \frac{\partial E}{\partial q_i} = \text{constant}$$

and the p's $(\overline{l}, \overline{g}, \overline{h})$ are linear functions of the time.

Since the von Zeipel equations connecting E, S, and F contain only the new variables, it is convenient to omit the bars and write simply (L, G, H) and (l, g, h) during the calculations. Of course, the bars must be restored before the explicit transformation equations and the new canonical equations can be written.

It is convenient to divide the discussion into two parts, since the equations of order 0, 1, and 2 are degenerate. It is also possible to omit many of the algebraic details by recalling that, for example, X_k is the only part of S_k that contains l, and that

$$\frac{\partial F_1}{\partial H} \frac{\partial Z_k}{\partial h} - \nu \frac{\partial Z_k}{\partial \lambda} = 0$$

since $F_1 = \nu H$ and Z_k contains h and λ only via their sum, $h + \lambda$.

<u>Initial stages</u>.- The first von Zeipel equation is simply

$$E_o = F_o = \frac{\mu^2}{2L^2}$$

and the second is

$$E_1 = F_1 + \frac{\partial F_o}{\partial L} \frac{\partial X_1}{\partial l} = \nu H - \frac{\mu^2}{L^3} \frac{\partial X_1}{\partial l}$$

which separates immediately into $X_1 = 0$, $E_1 = \nu H$, since F_1 is entirely of type E.

The equation of second order is, in view of the result just obtained,

$$E_2 = F_2 - \frac{\mu^2}{L^3}\frac{\partial X_2}{\partial l} + \nu\left(\frac{\partial Y_1}{\partial h} - \frac{\partial Y_1}{\partial \lambda}\right)$$

Since $ZF_2 = 0$, this equation separates into three:

$$\frac{\mu^2}{L^3}\frac{\partial X_2}{\partial l} = XF_2$$

$$\nu\left(\frac{\partial Y_1}{\partial \lambda} - \frac{\partial Y_1}{\partial h}\right) = YF_2$$

$$E_2 = EF_2$$

and integrating gives

$$X_2 = \frac{1}{4}\nu_2\frac{L^3}{\mu^2}\left[A_1 + 3B_1\cos(2g + 2h) - 3C_1\sin(2g + 2h)\right]$$

$$Y_1 = (\phi - \lambda)H - \frac{15}{16}\nu_1 a^2 e^2 \sin(2g + 2h)$$

$$E_2 = \frac{1}{4}\nu_2 a^2\left(1 + \frac{3}{2}e^2\right) = \frac{1}{8}\frac{\nu_2}{\mu^2}(5L^4 - 3G^2L^2)$$

where

$$A_1 = \int A_o\,dl = \int A_o(1 - e\cos u)du$$
$$= a^2\left[\left(\frac{3}{4}e^3 - 2e\right)\sin u + \frac{3}{4}e^2\sin 2u - \frac{1}{12}e^3\sin 3u\right]$$

$$B_1 = \int B_o\,dl$$
$$= a^2\left[\left(\frac{5}{4}e^3 - \frac{5}{2}e\right)\sin u + \left(\frac{1}{2} + \frac{1}{4}e^2\right)\sin 2u + \left(\frac{1}{12}e^3 - \frac{1}{6}e\right)\sin 3u\right]$$

$$C_1 = \int C_o\,dl$$
$$= a^2\sqrt{1 - e^2}\left[\frac{5}{2}e\left(\frac{1}{2}e + \cos u\right) - \frac{1}{2}(1 + e^2)\cos 2u + \frac{1}{6}e\cos 3u\right]$$

Following Hori (ref. 2) the term $(1/2)e$ is added to $\cos u$ to annihilate the mean value with respect to l.

The general case, order $k \geq 3$.- When the explicit equation for the new Hamiltonian was separated according to the order k, it was implicitly assumed that the order of a term is not affected by the process of differentiation.

However, the present Hamiltonian, beginning with F_3 contains the factor $\gamma = 1 - (H/G)$, so that differentiation may "lose" one power of γ, thereby reducing the order. For example, the terms

$$\frac{\partial F_3}{\partial \gamma}\frac{\partial \gamma}{\partial G} \quad \text{and} \quad \frac{\partial F_3}{\partial \gamma}\frac{\partial \gamma}{\partial H}$$

are of second order. When this is taken into account, the general equation, of order k can be written in the form

$$E_k = -\frac{\mu^2}{L^3}\frac{\partial X_k}{\partial l} + \nu\left(\frac{\partial Y_{k-1}}{\partial h} - \frac{\partial Y_{k-1}}{\partial \lambda}\right) + \left(\frac{\partial F_2}{\partial G} + \frac{\partial F_3}{\partial \gamma}\frac{\partial \gamma}{\partial G}\right)\frac{\partial Z_{k-2}}{\partial g}$$

$$+ \left(\frac{\partial F_2}{\partial H} + \frac{\partial F_3}{\partial \gamma}\frac{\partial \gamma}{\partial H}\right)\frac{\partial Z_{k-2}}{\partial h} + Q_k$$

where Q_k is a sum of terms involving F_k and previously computed quantities. The currently undetermined components X_k, Y_{k-1}, Z_{k-2} occur only in the terms exhibited here explicitly.

Recalling the separation of F_2 and F_3 into components, it can be seen that the terms

$$\left(\frac{\partial XF_2}{\partial G} + \frac{\partial XF_3}{\partial \gamma}\frac{\partial \gamma}{\partial G}\right)\frac{\partial Z_{k-2}}{\partial g} + \left(\frac{\partial XF_2}{\partial H} + \frac{\partial XF_3}{\partial \gamma}\frac{\partial \gamma}{\partial H}\right)\frac{\partial Z_{k-2}}{\partial h}$$

are entirely of type X, while the terms

$$\left(\frac{\partial YF_2}{\partial G} + \frac{\partial YF_3}{\partial \gamma}\frac{\partial \gamma}{\partial G}\right)\frac{\partial Z_{k-2}}{\partial g} + \left(\frac{\partial YF_2}{\partial H} + \frac{\partial YF_3}{\partial \gamma}\frac{\partial \gamma}{\partial H}\right)\frac{\partial Z_{k-2}}{\partial h}$$

are entirely of type Y. Since ZF_3 contains the factor e^2 as well as $\gamma\nu_2$, the terms containing $\partial ZF_3/\partial\gamma$ and Z_{k-2} can be regarded as of order $k + 1$, under the assumption that $e^2 = O(m)$; that is, these terms are deferred to the next stage. Since $ZF_2 = (\partial EF_2/\partial H) = 0$, the Z component of the equation for E_k becomes simply

$$\left(\frac{\partial EF_2}{\partial G} + \frac{\partial EF_3}{\partial \gamma}\frac{\partial \gamma}{\partial G}\right)\frac{\partial Z_{k-2}}{\partial g} + \frac{\partial EF_3}{\partial \gamma}\frac{\partial \gamma}{\partial H}\frac{\partial Z_{k-2}}{\partial h} + ZQ_k = 0$$

Inserting EF_2 and EF_3 gives

$$\xi\,\frac{\partial Z_{k-2}}{\partial g} + \eta\,\frac{\partial Z_{k-2}}{\partial h} = ZQ_k$$

where

$$\xi = \frac{3}{4} \nu_2 \frac{a^2}{G} \left[1 - e^2 + (1 - \gamma)\left(1 + \frac{3}{2} e^2\right)\right]$$

$$\eta = -\frac{3}{4} \nu_2 \frac{a^2}{G} \left(1 + \frac{3}{2} e^2\right)$$

If the typical term of ZQ_k has the form $f(\theta)$ with $\theta = ig + j(h + \lambda)$, then the corresponding term in Z_{k-2}, is

$$\frac{1}{\xi i + \eta j} \int f(\theta) d\theta$$

The only case that can produce a small denominator is $j = 2i$, giving

$$\xi i + \eta j = -\frac{3}{4} \nu_2 \frac{a^2 i}{G} \left[\frac{5}{2} e^2 + \gamma\left(1 + \frac{3}{2} e^2\right)\right]$$

If γ vanishes, the orbit plane coincides with the ecliptic, the node loses its identity, and g, h, occur only in the sum $g + h$. Hence, terms of type Z must contain the single argument, $g + h + \lambda$, so that $j = i$ and the small divisor does not occur. Hence, the divisor never vanishes.

Once Z_{k-2} has been obtained, the equation for E_k separates into three:

$$\frac{\mu^2}{L^3} \frac{\partial X_k}{\partial l} = \text{sum of terms of type } X$$

$$\nu \left(\frac{\partial Y_{k-1}}{\partial \lambda} - \frac{\partial Y_{k-1}}{\partial h}\right) = \text{sum of terms of type } Y$$

$$E_k = \text{sum of terms of type } E$$

and these can be integrated by quadrature.

The equations of order 3 and 4.- If the notation of the preceding section is used, the known terms of order three are

$$Q_3 = F_3 + \nu\left(\frac{\partial X_2}{\partial h} - \frac{\partial X_2}{\partial \lambda}\right) + \left(\frac{\partial F_2}{\partial G} + \frac{\partial F_3}{\partial \gamma} \frac{\partial \gamma}{\partial G}\right) \frac{\partial Y_1}{\partial g} + \left(\frac{\partial F_2}{\partial H} + \frac{\partial F_3}{\partial \gamma} \frac{\partial \gamma}{\partial H}\right) \frac{\partial Y_1}{\partial h}$$

First consider the terms involving F_3 and Y_1. These can be put in the form

$$\frac{1}{G} \frac{\partial F_3}{\partial \gamma} \left(\frac{\partial Y_1}{\partial g} - \frac{\partial Y_1}{\partial h}\right) - \frac{\gamma}{G} \frac{\partial F_3}{\partial \gamma} \frac{\partial Y_1}{\partial g}$$

by noting that

$$\gamma = 1 - \frac{H}{G} \qquad \frac{\partial\gamma}{\partial G} = \frac{H}{G^2} = \frac{1-\gamma}{G}$$

$$H = G(1 - \gamma) \qquad \frac{\partial\gamma}{\partial H} = -\frac{1}{G}$$

The second term, due to the reappearance of the "lost" γ, is of fourth order, and hence can be deferred to the next stage (this is true at every stage). The first term vanishes, since Y_1 contains the single argument $g + h$ (this does not occur at later stages).

Next consider the term

$$\frac{\partial F_2}{\partial H}\frac{\partial Y_1}{\partial h} = -\frac{15}{8}\nu_2\delta a^2e^2 \cos(2g + 2h)$$

Again the series expansion for the solar factor δ must be used, giving

$$-\frac{15}{16}\nu_2\delta_1 a^2e^2[\cos(2g + 2h + \lambda) + \cos(2g + 2h - \lambda)]$$

$$-\frac{15}{16}\nu_2\delta_2 a^2e^2[\cos(2g + 2h + 2\lambda) + \cos(2g + 2h - 2\lambda)]$$

$$+ \ldots$$

The terms on the first line are of third order, while the remaining terms can be deferred to later stages. The treatment of the other terms in Q_3 is straightforward. In particular, the only term of type Z is

$$ZQ_3 = ZF_3 = \frac{15}{8}\gamma\nu_2 a^2e^2 \cos 2g$$

and the general theory of the preceding section, with $i = 2$, $j = 0$, gives

$$Z_1 = \frac{\frac{5}{8}\gamma G e^2 \sin 2g}{1 + \frac{1}{4}e^2 - \frac{1}{2}\gamma\left(1 + \frac{3}{2}e^2\right)}$$

Expanding the denominator by the binomial theorem, with

$$\zeta = \frac{1}{2}\gamma\left(1 + \frac{3}{2}e^2\right) - \frac{1}{4}e^2$$

gives

$$Z_1 = \frac{5}{8} \gamma G e^2 \sin 2g$$

and the remaining terms

$$Z_1 (\zeta + \zeta^2 + \zeta^3 + \ldots)$$

can be assigned to the higher stages $Z_2, Z_3, \ldots$.

The process of inserting Z_1 in the terms not used in its calculation is straightforward. In particular, the deferred terms involving $\partial ZF_3/\partial\gamma$ become

$$\frac{\partial ZF_3}{\partial \gamma} \frac{\partial \gamma}{\partial G} \frac{\partial Z_1}{\partial g} = \frac{75}{64} \gamma \nu_2 a^2 e^4 (1 - \gamma)(1 + \cos 4g)$$

thus making contributions of types Z and E to the fourth-order equation.

The rest of the procedure is straightforward, provided that wherever H/G occurs it is to be replaced by $1 - \gamma$ and the terms with the extra γ are reassigned to the next stage. Then the third-order equation yields

$$\begin{aligned}
X_3 = {} & \beta X_2 + \frac{3}{2} \gamma \nu_2 \frac{L^3}{\mu^2} \sin h \left(1 + \frac{5}{4} e^2 \cos 2g\right) [-A_1 \sin h \\
& + B_1 \sin(2g + h) + C_1 \cos(2g + h)] \\
& - \frac{3}{2} \nu \nu_2 \frac{L^6}{\mu^4} [B_2 \sin(2g + 2h) + C_2 \cos(2g + 2h)] \\
& + \frac{1}{4} \nu_2 \frac{L^3}{\mu^2} \left[\frac{5}{4} \gamma e^2 G \cos 2g - \frac{15}{8} \nu_1 a^2 e^2 \cos(2g + 2h)\right] \\
& \times \left[\frac{\partial A_1}{\partial G} + 3 \frac{\partial B_1}{\partial G} \cos(2g + 2h) - 3 \frac{\partial C_1}{\partial G} \sin(2g + 2h)\right]
\end{aligned}$$

$$\begin{aligned}
Y_2 = {} & \frac{1}{4} \nu_1 a^2 \left(1 + \frac{3}{2} e^2\right) (\phi - \lambda + \varepsilon \sin \phi) \\
& - \frac{45}{64} \nu_1{}^2 a^4 e^2 \frac{G}{L^2} \left[\sin(2g + 2h) + \frac{5}{4} \sin(4g + 4h)\right] \\
& + \frac{15}{16} \nu_1 (\delta_1 - \beta_1) a^2 e^2 \left[\sin(2g + 2h + \lambda) + \frac{1}{3} \sin(2g + 2h - \lambda)\right] \\
& - \frac{3}{8} \gamma \nu_1 a^2 \left(1 - \frac{13}{8} e^2 + \frac{25}{16} e^4\right) \sin 2h \\
& + \frac{45}{128} \gamma \nu_1 a^2 e^2 (2 - e^2) \sin(2g + 2h) + \frac{15}{64} \gamma \nu_1 a^2 e^2 \left(1 + \frac{3}{2} e^2\right) \sin(2g - 2h) \\
& + \frac{75}{128} \gamma \nu_1 a^2 e^2 (2 - e^2) \sin(4g + 2h)
\end{aligned}$$

$$E_3 = \frac{225}{64} \nu_1\nu_2 a^4 e^2 \frac{G}{L^2} - \frac{3}{4} \gamma\nu_2 a^2\left(1 + \frac{3}{2} e^2\right)$$

where

$$B_2 = \int B_1 \, dl \quad \text{and} \quad C_2 = \int C_1 \, dl$$

Partial results from the fourth stage are

$$\begin{aligned} Z_2 = {} & \frac{5}{8} Ge^2(\beta_2 - \delta_2 + \delta_1{}^2 - \delta_1\beta_1)\sin(2g + 2h + 2\lambda) \\ & + \left[- \frac{5}{32} \gamma Ge^4 + \frac{5}{4} \gamma^2 Ge^2\left(1 - \frac{75}{64} e^2 + \frac{75}{256} e^4\right)\right. \\ & \left. - \frac{45}{256} \gamma\nu_1 a^2 e^2(12 - 23e^2 + 11e^4)\right]\sin 2g \\ & + \left[\frac{25}{128} \gamma Ge^4 - \frac{25}{1024} \gamma^2 Ge^2(16 - 16e^2 + e^4)\right. \\ & \left. + \frac{225}{1024} \gamma\nu_1 a^2 e^4(1 - e^2)\right]\sin 4g \\ & - \frac{125}{3072} \gamma^2 e^4(4 - e^2)\sin 6g \end{aligned}$$

$$\begin{aligned} E_4 = {} & \frac{\nu_2{}^2 a^6}{L^2}\left(- \frac{49}{64} + \frac{873}{64} e^2 - \frac{4347}{512} e^4\right) + \frac{675}{1024} \nu_1{}^2\nu_2 \frac{a^6 e^2}{L^2} (4 - 5e^2) \\ & + \frac{9}{512} \frac{\gamma\nu_1\nu_2}{G} a^4(16 - 352e^2 + 411e^4 - 75e^6) + \frac{75}{64} \gamma\nu_2 a^2 e^4 \\ & + \frac{3}{512} \gamma^2\nu_2 a^2(64 - 304e^2 + 400e^4 - 25e^6) \end{aligned}$$

This completes the discussion of the lunar theory. The validity of the method has been established, and the calculations have been carried far enough to illustrate the manipulative techniques that are required.

ARTIFICIAL SATELLITE THEORY

Development of the Hamiltonian

The primary purpose here is to demonstrate that the generalized von Zeipel transformation can be applied directly, in a simple, degenerate form, to artificial satellite theory. The usual assumptions about the gravitational field of a triaxial ellipsoid give the Hamiltonian

$$F = \frac{\mu^2}{2L^2} + \nu H + \sum_{n=2}^{\infty} D_n + \sum_{n=2}^{\infty} \sum_{m=1}^{n} T_{n,m}$$

where the zonal harmonics, D_n, and the tesseral harmonics, $T_{n,m}$ are

$$D_n = -\mu J_n \frac{R^n}{r^{n+1}} P_n(\sin \theta)$$

$$T_{n,m} = \mu \frac{R^n}{r^{n+1}} (C_{n,m} \cos m\Lambda + S_{n,m} \sin m\Lambda) P_n{}^m(\sin \theta)$$

Here ν is the spin velocity of the earth, R is the equatorial radius of the earth, θ and Λ are the geographical latitude and longitude (measured east from Greenwich), J, C, and S are dimensionless constants, P_n is the Legendre polynomial, and $P_n{}^m$ is the associated Legendre function

$$P_n{}^m(Z) = (1 - Z^2)^{m/2} \frac{d^m P_n(Z)}{dZ^m}$$

The notation for the zonal harmonics is Brouwer's (ref. 1) and the notation for the tesseral harmonics is Izsak's (ref. 7). As in the lunar theory the term νH is added to the Hamiltonian because the longitude of the node, h, is measured from a rotating reference line (the meridian of Greenwich).

Orders of magnitude can be defined by introducing n, the mean motion of the satellite, the dimensionless parameter $m = \nu/n << 1$, and the usual assumptions (Garfinkel, ref. 8)

$$J_2 = 0(m^2)$$

$$J_n = 0(J_2{}^2) \;, \qquad n > 2$$

$$C_{n,m} = 0(J_2{}^2)$$

$$S_{n,m} = 0(J_2{}^2)$$

Thus,

$$F = F_o + F_1 + F_2 + F_3 + F_4$$

$$F_o = \frac{\mu^2}{2L^2}$$

$$F_1 = \nu H$$

$$F_2 = D_2$$

$$F_3 = 0$$

F_4 = sum of tesseral harmonics and remaining zonal harmonics

To decompose F_k by types (X, Y, Z, E)

$$F_k = XF_k + YF_k + ZF_k + EF_k$$

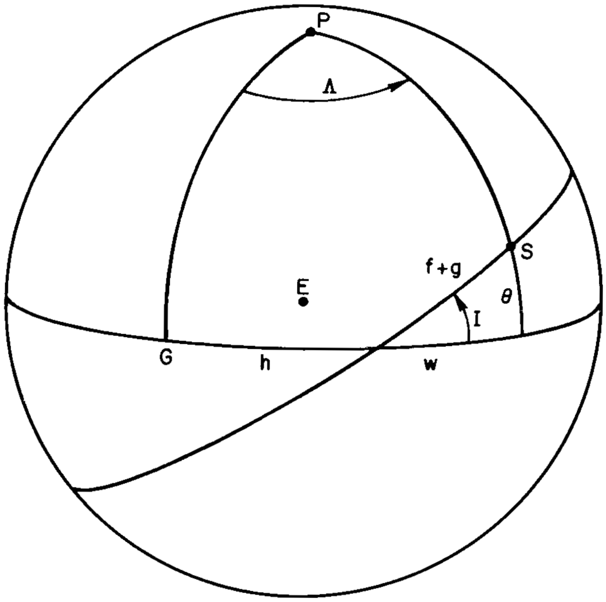
note that, since the present Hamiltonian does not contain the time, the definitions reduce to

X: periodic in l

Y: periodic in g and h, independent of l

Z: periodic in g, independent of l and h

E: independent of all three angle variables



Following Brouwer (ref. 1), introduce l, the mean anomaly, implicitly via f, the true anomaly, by means of the formulas

$$r = \frac{G^2/\mu}{1 + e \cos f}$$

$$\sin \theta = \sin I \sin(f + g)$$

Then the Legendre polynomials become trigonometric polynomials in $f + g$. For example,

$$P_2(\sin \theta) = \frac{3}{2} \sin^2 \theta - \frac{1}{2} = \frac{1}{4} (1 - 3 \cos^2 I) - \frac{3}{4} \sin^2 I \cos(2f + 2g)$$

$$P_3(\sin \theta) = \frac{5}{2} \sin^3 \theta - \frac{3}{2} \sin \theta$$

$$= \frac{3}{8} \sin I(1 - 5 \cos^2 I)\sin(f + g) - \frac{5}{8} \sin^3 I \sin(3f + 3g)$$

$$P_4(\sin \theta) = \frac{35}{8} \sin^4 \theta - \frac{15}{4} \sin^2 \theta + \frac{3}{8}$$

$$= \frac{3}{64} (3 - 30 \cos^2 I + 35 \cos^4 I)$$

$$- \frac{5}{16} \sin^2 I(1 - 7 \cos^2 I)\cos(2f + 2g) + \frac{35}{64} \sin^4 I \cos(4f + 4g)$$

and the associated Legendre functions can be treated similarly, since

$$P_n^m(\sin\theta) = \cos^m\theta \, \frac{d^m P_n(\sin\theta)}{d(\sin\theta)^m}$$

Thus, for example,

$$P_2^2(\sin\theta) = 3\cos^2\theta$$

$$P_3^1(\sin\theta) = \cos\theta\left(\frac{15}{2}\sin^2\theta - \frac{3}{2}\right) = \frac{3}{4}\cos\theta[3 - 5\cos^2 I - 5\sin^2 I\cos(2f + 2g)]$$

$$P_3^2(\sin\theta) = 15\cos^2\theta\sin\theta = 15\cos^2\theta\sin I\sin(f + g)$$

$$P_3^3(\sin\theta) = 15\cos^3\theta$$

The factor $\cos^m\theta$ will disappear by cancellation once the longitudinal factors have been obtained. To show this, use the trigonometric formulas (ref. 9)

$$w = \Lambda - h$$

$$\sin w = \frac{\sin(f + g)\cos I}{\cos\theta}$$

$$\cos w = \frac{\cos(f + g)}{\cos\theta}$$

to obtain

$$\sin\Lambda = \frac{c_1\sin h + s_1\cos h}{\cos\theta}$$

$$\cos\Lambda = \frac{c_1\cos h - s_1\sin h}{\cos\theta}$$

$$c_1 = \cos(f + g)$$

$$s_1 = \cos I\sin(f + g)$$

and hence, by induction,

$$\sin m\Lambda = \frac{c_m \sin m h + s_m \cos m h}{\cos^m \theta}$$

$$\cos m\Lambda = \frac{c_m \cos m h - s_m \sin m h}{\cos^m \theta}$$

$$c_{m+1} = c_m c_1 - s_m s_1$$

$$s_{m+1} = c_m s_1 + s_m c_1$$

so that, for example,

$$c_2 = \frac{1}{2} \sin^2 I + \frac{1}{2} (1 + \cos^2 I)\cos(2f + 2g)$$

$$s_2 = \cos I \sin(2f + 2g)$$

$$c_3 = \frac{3}{4} \sin^2 I \cos(f + g) + \frac{1}{4} (1 + 3 \cos^2 I)\cos(3f + 3g)$$

$$s_3 = \frac{3}{4} \sin^2 I \cos I \sin(f + g) + \frac{1}{4} (3 + \cos^2 I)\cos I \sin(3f + 3g)$$

These yield the desired expressions for the tesseral harmonics, typical examples being

$$T_{2,2} = 3\mu \frac{R^2}{r^3} \left\{(C_{2,2} \cos 2h + S_{2,2} \sin 2h)\left[\frac{1}{2} \sin^2 I + \frac{1}{2} (1 + \cos^2 I)\cos(2f + 2g)\right]\right.$$
$$\left. + (S_{2,2} \cos 2h - C_{2,2} \sin 2h)\cos I \sin(2f + 2g)\right\}$$

$$T_{3,1} = \frac{3}{4} \mu \frac{R^3}{r^4} [(C_{3,1} \cos h + S_{3,1} \sin h)\cos(f + g)$$
$$+ (S_{3,1} \cos h - C_{3,1} \sin h)\cos I \sin(f + g)]$$
$$\times [3 - 5 \cos^2 I - 5 \sin^2 I \cos(2f + 2g)]$$

Thus, both zonal and tesseral harmonics can be expressed as trigonometric polynomials in $f + g$, multiplied by negative powers of r, and the coefficients are functions of I and h. To effect the decomposition by types, introduce the mean values with respect to l (Brouwer, ref. 1):

$$\bar{D}_n = \frac{1}{2\pi} \int_0^{2\pi} D_n \, dl$$

$$\bar{T}_{n,m} = \frac{1}{2\pi} \int_0^{2\pi} T_{n,m} \, dl$$

so that the components of type X are simply

$$XD_n = D_n - \bar{D}_n$$

$$XT_{n,m} = T_{n,m} - \bar{T}_{n,m}$$

Perform the integrations by changing to the true anomaly, f, as the variable of integration, using the formulas

$$dl = \frac{\mu^2 r^2}{GL^3} \, df, \qquad r = \frac{G^2/\mu}{1 + e \cos f}$$

Then the negative powers of r become positive powers of $1 + e \cos f$, and the integrands become trigonometric polynomials in f.

The zonal harmonics have no components of type Y, and the tesseral harmonics have no components of type Z or E. Hence,

$$\bar{D}_n = ZD_n + ED_n$$

$$\bar{T}_{n,m} = YT_{n,m}$$

and the separation of $\bar{D}_n$ is done by inspection. Typical results are:

$$F_2 = -\mu J_2 \frac{R^2}{r^3} P_2(\sin \theta) = \frac{1}{4} \mu J_2 \frac{R^2}{r^3} [3 \cos^2 I - 1 + 3 \sin^2 I \cos(2f + 2g)]$$

$$EF_2 = \bar{F}_2 = \frac{1}{4} J_2 \frac{\mu^4 R^2}{G^3 L^3} (3 \cos^2 I - 1)$$

$$YF_2 = ZF_2 = 0, \qquad XF_2 = F_2 - EF_2$$

The third zonal harmonic yields

$$ZD_3 = \bar{D}_3 = \frac{3}{8} J_3 \frac{\mu^5 R^3 e}{G^5 L^3} \sin I (5 \cos^2 I - 1) \sin g$$

$$YD_3 = ED_3 = 0, \qquad XD_3 = D_3 - ZD_3$$

and the fourth gives

$$ZD_4 = \frac{15}{64} J_4 \frac{\mu^6 R^4 e^2}{G^7 L^3} \sin^2 I(1 - 7 \cos^2 I)\cos 2g$$

$$ED_4 = - \frac{3}{64} J_4 \frac{\mu^6 R^4}{G^7 L^3} \left(1 + \frac{3}{2} e^2\right)(3 - 30 \cos^2 I + 35 \cos^4 I)$$

$$YD_4 = 0, \qquad XD_4 = D_4 - (ZD_4 + ED_4)$$

The first two tesseral harmonics give

$$YT_{2,2} = \frac{3}{2} \frac{\mu^4 R^2}{G^3 L^3} \sin^2 I(C_{2,2} \cos 2h + S_{2,2} \sin 2h)$$

$$XT_{2,2} = T_{2,2} - YT_{2,2}$$

and

$$YT_{3,1} = \frac{3}{8} \frac{\mu^5 R^3 e}{G^5 L^3} [(C_{3,1} \cos h + S_{3,1} \sin h)(1 - 5 \cos^2 I)\cos g$$

$$+ (S_{3,1} \cos h - C_{3,1} \sin h)(11 - 15 \cos^2 I)\sin g]$$

It is clear that every harmonic can be decomposed in this fashion.

Solution of the Artificial Satellite Problem

Initial stages.- The generalized von Zeipel procedure gives, for the first three stages, essentially the first-order results obtained by Brouwer (ref. 1).

$$E_o = F_o = \frac{\mu^2}{2L^2}$$

$$E_1 = F_1 - \frac{\mu^2}{L^3} \frac{\partial X_1}{\partial l} = F_1 = \nu H$$

and, as in the lunar theory, $X_1 = 0$. The second-order equation is

$$E_2 = XF_2 + EF_2 - \frac{\mu^2}{L^3} \frac{\partial X_2}{\partial l} + \nu \frac{\partial Y_1}{\partial h}$$

Separating and integrating gives

$$Y_1 = 0$$

$$E_2 = EF_2 = \frac{1}{4} J_2 \frac{\mu^4 R^2}{G^3 L^3} \left(3 \frac{H^2}{G^2} - 1\right)$$

$$X_2 = \frac{L^3}{\mu^2} \int XF_2 \, dl = \frac{L^3}{\mu^2} \left(\int F_2 \, dl - E_2 l\right)$$

$$= \frac{1}{4} J_2 \frac{\mu^2 R^2}{G^3} \left\{(3 \cos^2 I - 1)(f - l + e \sin f)\right.$$

$$\left. + 3 \sin^2 I\left[\frac{1}{2} \sin(2f + 2g) + \frac{1}{2} e \sin(f + 2g) + \frac{1}{6} e \sin(3f + 2g)\right]\right\}$$

The general case, order $k \geq 3$.- Since the quantities λ and γ of the lunar theory do not occur in the satellite problem, the von Zeipel equations do not suffer any loss of order due to differentiation. Hence, the general equation becomes

$$E_k = - \frac{\mu^2}{L^3} \frac{\partial X_k}{\partial l} + \nu \frac{\partial Y_{k-1}}{\partial h} + \frac{\partial F_2}{\partial G} \frac{\partial Z_{k-2}}{\partial g} + Q_k$$

where Q_k is a sum of known terms. The component Z_{k-2} is obtained from

$$\frac{\partial EF_2}{\partial G} \frac{\partial Z_{k-2}}{\partial g} + ZQ_k = 0$$

that is,

$$\frac{3}{4} J_2 \frac{\mu^4 R^2}{G^4 L^3} (5 \cos^2 I - 1) \frac{\partial Z_{k-2}}{\partial g} = ZQ_k$$

which can be integrated by quadrature. The general equation then becomes

$$E_k = - \frac{\mu^2}{L^3} \frac{\partial X_k}{\partial l} + \nu \frac{\partial Y_{k-1}}{\partial h} + \frac{\partial XF_2}{\partial G} \frac{\partial Z_{k-2}}{\partial g} + Q_k - ZQ_k$$

which separates into three:

$$\frac{\mu^2}{L^3} \frac{\partial X_k}{\partial l} = XQ_k + \frac{\partial XF_2}{\partial G} \frac{\partial Z_{k-2}}{\partial g}$$

$$-\nu \frac{\partial Y_{k-1}}{\partial h} = YQ_k$$

$$E_k = EQ_k$$

Notice that the partial differential equations of lunar theory have degenerated to ordinary equations here.

Equations of order 3 and 4.- The third-order equation is completely degenerate since the known terms are

$$Q_3 = F_3 + \nu \frac{\partial X_2}{\partial h} + \frac{\partial F_2}{\partial G}\left(\frac{\partial X_1}{\partial g} + \frac{\partial Y_1}{\partial g}\right) + \frac{\partial F_2}{\partial L}\frac{\partial X_1}{\partial l} + \frac{\partial^2 F_0}{\partial L^2}\frac{\partial X_1}{\partial l}\frac{\partial X_2}{\partial l} = 0$$

giving

$$E_3 = X_3 = Y_2 = Z_1 = 0$$

For the fourth-order equation the known terms are

$$Q_4 = F_4 + \frac{\partial F_2}{\partial L}\frac{\partial X_2}{\partial l} + \frac{\partial F_2}{\partial G}\frac{\partial X_2}{\partial g} + \frac{1}{2}\frac{\partial^2 F_0}{\partial L^2}\left(\frac{\partial X_2}{\partial l}\right)^2$$

Since this expression is quite lengthy the components of type X will not be exhibited. The zonal harmonics give the components of types Z and E which yield Brouwer's results (ref. 1)

$$Z_2 = \frac{1}{32}\frac{\mu^2 R^2 e^2}{G^3}\frac{\sin^2 I}{5\cos^2 I - 1}\left[J_2(1 - 15\cos^2 I)\right.$$

$$\left. + \frac{5J_4}{J_2}(1 - 7\cos^2 I)\right]\sin 2g - \frac{1}{2}\frac{J_3}{J_2}\frac{\mu Re}{G}\sin I\cos g$$

$$E_4 = \frac{3}{128}J_2{}^2\frac{\mu^6 R^4}{G^7 L^3}\left[\frac{G^2}{L^2}(5\cos^4 I - 18\cos^2 I + 5)\right.$$

$$\left. + 4\frac{G}{L}(3\cos^2 I - 1)^2 + 5(7\cos^4 I + 2\cos^2 I - 1)\right]$$

$$+ \frac{3}{128}J_4\frac{\mu^6 R^4}{G^7 L^3}\left(3\frac{G^2}{L^2} - 5\right)(35\cos^4 I - 30\cos^2 I + 3)$$

The first two tesseral harmonics yield the typical contribution to Y_3:

$$Y_3 = -\frac{3}{4}\frac{\mu^4 R^2}{\nu G^3 L^3}\sin^2 I(C_{2,2}\sin 2h - S_{2,2}\cos 2h)$$

$$+ \frac{3}{8}\frac{\mu^5 R^3 e}{\nu G^5 L^3}[(5\cos^2 I - 1)\cos g(C_{3,1}\sin h - S_{3,1}\cos h)$$

$$+ (15\cos^2 I - 11)\sin g(S_{3,1}\sin h + C_{3,1}\cos h)]$$

Again, the development can be terminated with the remark that the validity of the method has been established, provided the critical inclination is avoided.

CONCLUDING REMARKS

The generalized von Zeipel transformation presented here is clearly capable of producing a highly efficient lunar theory. The major simplification resides in the partial uncoupling of solar and lunar factors. This occurs in the initial development of the Hamiltonian and in the final development of the determining function. At both stages many Fourier series expansions are avoided, the result being a product of two finite expressions. This is achieved primarily because the elimination of the short period terms is effected by solving an ordinary differential equation, in which solar factors play the role of constants.

If the inclination angle is not restricted to small values, the same technique can be applied to eliminate the terms of types X and Y, with terms of type Z remaining in the new Hamiltonian. The resulting system might well be a fruitful subject for future research.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., July 24, 1969

APPENDIX

EXPLICIT RECURSIVE ALGORITHMS FOR THE VON ZEIPEL TRANSFORMATION

The basic, implicit equations of the von Zeipel transformation are

$$x_i = q_i + \frac{\partial S(q, y)}{\partial y_i}$$

$$p_i = y_i + \frac{\partial S(q, y)}{\partial q_i}$$

where, for present purposes, the dependence on t, via λ, is irrelevant. Let $\phi(x, y)$ be an arbitrary function and let

$$\psi(q, p) = \phi(x, y)$$

That is, this is an identity in either set of variables under the transformation $(x, y) \leftrightarrow (q, p)$. Eliminating p in the left member and x in the right member gives

$$\psi\left\{q, \left[y_i + \frac{\partial S(q, y)}{\partial q_i}\right]\right\} = \phi\left\{\left[q_i + \frac{\partial S(q, y)}{\partial y_i}\right], y\right\}$$

an identity in (q, y). Expanding each member in Taylor's series gives

$$\psi(q, y) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \psi(q, y)}{\partial y_{i_1} \partial y_{i_2} \cdots \partial y_{i_k}} \frac{\partial S}{\partial q_{i_1}} \frac{\partial S}{\partial q_{i_2}} \cdots \frac{\partial S}{\partial q_{i_k}}$$

$$= \phi(q, y) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \phi(q, y)}{\partial q_{i_1} \partial q_{i_2} \cdots \partial q_{i_k}} \frac{\partial S}{\partial y_{i_1}} \frac{\partial S}{\partial y_{i_2}} \cdots \frac{\partial S}{\partial y_{i_k}}$$

with the repeated subscripts $i_1, i_2, \ldots, i_k$ satisfying the summation convention. Since this is an identity in the variables (q, y), the variables are dummies, and can be replaced by (q, p) or by (x, y). Introducing the series expansion

$$\phi = \sum_{n=o}^{\infty} \phi_n$$

$$\psi = \sum_{n=o}^{\infty} \psi_n$$

$$S = \sum_{n=1}^{\infty} S_n$$

and collecting terms of equal order gives, with (q, y) replaced by (q, p):

$$\psi_o(q, p) = \phi_o(q, p)$$

and, for $n \geq 1$,

$$\psi_n(q, p) + \sum_{k=1}^{n} \frac{1}{k!} \sum_{\alpha} \frac{\partial^k \psi_\alpha(q, p)}{\partial p_{i_1}\, \partial p_{i_2} \cdots \partial p_{i_k}} \frac{\partial S_{\alpha_1}(q, p)}{\partial q_{i_1}} \cdots \frac{\partial S_{\alpha_k}(q, p)}{\partial q_{i_k}}$$

$$= \phi_n(q, p) + \sum_{k=1}^{n} \frac{1}{k!} \sum_{\alpha} \frac{\partial^k \phi_\alpha(q, p)}{\partial q_{i_1}\, \partial q_{i_2} \cdots \partial q_{i_k}} \frac{\partial S_{\alpha_1}(q, p)}{\partial p_{i_1}} \cdots \frac{\partial S_{\alpha_k}(q, p)}{\partial p_{i_k}}$$

where $\sum\limits_{\alpha}$ denotes the sum over all combinations of $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_k$ satisfying

$$\alpha \geq 0, \qquad \alpha_j \geq 1$$

$$\alpha + \sum_{j=1}^{k} \alpha_j = n$$

This can be put in the explicit form

$$\psi_n + \sum_{k=1}^{n} \frac{1}{k!} \sum_{\alpha_1=1}^{r_1} \sum_{\alpha_2=1}^{r_2} \cdots \sum_{\alpha_k=1}^{r_k} \frac{\partial^k \psi_\alpha}{\partial p_{i_1}\, \partial p_{i_2} \cdots \partial p_{i_k}} \frac{\partial S_{\alpha_1}}{\partial q_{i_1}} \cdots \frac{\partial S_{\alpha_k}}{\partial q_{i_k}}$$

$$= \phi_n + \sum_{k=1}^{n} \frac{1}{k!} \sum_{\alpha_1=1}^{r_1} \sum_{\alpha_2=1}^{r_2} \cdots \sum_{\alpha_k=1}^{r_k} \frac{\partial^k \phi_\alpha}{\partial q_{i_1}\, \partial q_{i_2} \cdots \partial q_{i_k}} \frac{\partial S_{\alpha_1}}{\partial p_{i_1}} \cdots \frac{\partial S_{\alpha_k}}{\partial p_{i_k}}$$

where

$$r_1 = n + 1 - k$$

$$r_{j+1} = r_j - \alpha_j + 1 \quad \text{for } j = 1, 2, \ldots, k-1$$

$$\alpha = r_k - \alpha_k$$

Thus, for example,

$$\psi_o = \phi_o$$

$$\psi_1 + \frac{\partial\psi_o}{\partial p_i}\frac{\partial S_1}{\partial q_i} = \phi_1 + \frac{\partial\phi_o}{\partial q_i}\frac{\partial S_1}{\partial p_i}$$

$$\psi_2 + \frac{\partial\psi_1}{\partial p_i}\frac{\partial S_1}{\partial q_i} + \frac{\partial\psi_o}{\partial p_i}\frac{\partial S_2}{\partial q_i} + \frac{1}{2}\frac{\partial^2\psi_o}{\partial p_i\ \partial p_j}\frac{\partial S_1}{\partial q_i}\frac{\partial S_1}{\partial q_j}$$

$$= \phi_2 + \frac{\partial\phi_1}{\partial q_i}\frac{\partial S_1}{\partial p_i} + \frac{\partial\phi_o}{\partial q_i}\frac{\partial S_2}{\partial p_i} + \frac{1}{2}\frac{\partial^2\phi_o}{\partial q_i\ \partial q_j}\frac{\partial S_1}{\partial p_i}\frac{\partial S_1}{\partial p_j}$$

and so forth, a recursive system. Thus, if $\phi_n(x, y)$ are given, then $\psi_n(q, p)$ are obtained recursively. Similarly, if these same equations are written with (x, y) in place of (q, p), then $\phi_n(x, y)$ can be obtained recursively when $\psi_n(q, p)$ are given. The equations for transforming the variables (x, y) and (q, p), in either direction, now follow immediately as special cases, by taking, successively

$$\phi_o(x, y) = x_\beta, \qquad \phi_\alpha = 0 \qquad \text{for} \quad \alpha \neq 0$$

$$\phi_o(x, y) = y_\beta, \qquad \phi_\alpha = 0 \qquad \text{for} \quad \alpha \neq 0$$

$$\psi_o(q, p) = q_\beta, \qquad \psi_\alpha = 0 \qquad \text{for} \quad \alpha \neq 0$$

$$\psi_o(q, p) = p_\beta, \qquad \psi_\alpha = 0 \qquad \text{for} \quad \alpha \neq 0$$

The results are

$$x_\beta = \sum_{n=o}^{\infty} q_{\beta,n}$$

where

$$q_{\beta,o} = q_\beta$$

$$q_{\beta,1} = \frac{\partial S_1}{\partial p_\beta}$$

and, for $n \geq 2$,

$$q_{\beta,n} = \frac{\partial S_n}{\partial p_\beta} - \sum_{k=1}^{n-1}\frac{1}{k!}\sum_{\alpha_1=1}^{t_1} \cdots \sum_{\alpha_k=1}^{t_k} \frac{\partial^k q_{\beta,\alpha}}{\partial p_{i_1} \cdots \partial p_{i_k}} \frac{\partial S_{\alpha_1}}{\partial q_{i_1}} \cdots \frac{\partial S_{\alpha_k}}{\partial q_{i_k}}$$

where

$$t_1 = n - k$$

$$t_{j+1} = t_j - \alpha_j + 1 \qquad \text{for} \quad j = 1, 2, \ldots, k - 1$$

$$\alpha = t_k - \alpha_k + 1$$

Thus, for example,

$$q_{\beta,2} = \frac{\partial S_2}{\partial p_\beta} - \frac{\partial q_{\beta,1}}{\partial p_i} \frac{\partial S_1}{\partial q_i}$$

$$q_{\beta,3} = \frac{\partial S_3}{\partial p_\beta} - \frac{\partial q_{\beta,1}}{\partial p_i} \frac{\partial S_2}{\partial q_i} - \frac{\partial q_{\beta,2}}{\partial p_i} \frac{\partial S_1}{\partial q_i} - \frac{1}{2} \frac{\partial^2 q_{\beta,1}}{\partial p_i \, \partial p_j} \frac{\partial S_1}{\partial q_i} \frac{\partial S_1}{\partial q_j}$$

and so forth, where $S_k = S_k(q, p)$.

The transformation equation for y is

$$y_\beta = \sum_{n=o}^{\infty} p_{\beta,n}$$

where

$$p_{\beta,o} = p_\beta$$

$$p_{\beta,1} = - \frac{\partial S_1}{\partial q_\beta}$$

and, for $n \geq 2$,

$$p_{\beta,n} = - \sum_{k=1}^{n-1} \frac{1}{k!} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_k=1}^{r_k} \frac{\partial^k p_{\beta,\alpha}}{\partial p_{i_1} \cdots \partial p_{i_k}} \frac{\partial S_{\alpha_1}}{\partial q_{i_1}} \cdots \frac{\partial S_{\alpha_k}}{\partial q_{i_k}}$$

Thus, for example,

$$p_{\beta,2} = - \frac{\partial S_2}{\partial q_\beta} - \frac{\partial p_{\beta,1}}{\partial p_i} \frac{\partial S_1}{\partial q_i}$$

$$p_{\beta,3} = - \frac{\partial S_3}{\partial q_\beta} - \frac{\partial p_{\beta,1}}{\partial p_i} \frac{\partial S_2}{\partial q_i} - \frac{\partial p_{\beta,2}}{\partial p_i} \frac{\partial S_1}{\partial q_i} - \frac{1}{2} \frac{\partial^2 p_{\beta,1}}{\partial p_i \, \partial p_j} \frac{\partial S_1}{\partial q_i} \frac{\partial S_1}{\partial q_j}$$

and so forth, again with $S_k = S_k(q, p)$.

The reverse transformation is

$$q_\beta = \sum_{n=o}^{\infty} x_{\beta,n}$$

where

$$x_{\beta,o} = x_\beta$$

$$x_{\beta,1} = - \frac{\partial S_1}{\partial y_\beta}$$

and, for $n \geq 2$,

$$x_{\beta,n} = - \sum_{k=1}^{n-1} \frac{1}{k!} \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_k=1}^{r_k} \frac{\partial^k x_{\beta,\alpha}}{\partial x_{i_1} \cdots \partial x_{i_k}} \frac{\partial S_{\alpha_1}}{\partial y_{i_1}} \cdots \frac{\partial S_{\alpha_k}}{\partial y_{i_k}}$$

Thus, for example,

$$x_{\beta,2} = - \frac{\partial S_2}{\partial y_\beta} - \frac{\partial x_{\beta,1}}{\partial x_i} \frac{\partial S_1}{\partial y_i}$$

$$x_{\beta,3} = - \frac{\partial S_3}{\partial y_\beta} - \frac{\partial x_{\beta,1}}{\partial x_i} \frac{\partial S_2}{\partial y_i} - \frac{\partial x_{\beta,2}}{\partial x_i} \frac{\partial S_1}{\partial y_i} - \frac{1}{2} \frac{\partial^2 x_{\beta,1}}{\partial x_i \, \partial x_j} \frac{\partial S_1}{\partial y_i} \frac{\partial S_1}{\partial y_j}$$

and so forth, where now $S_k = S_k(x, y)$.

Finally, the transformation equation for p is

$$p_\beta = \sum_{n=o}^{\infty} y_{\beta,n}$$

where

$$y_{\beta,o} = y_\beta$$

$$y_{\beta,1} = \frac{\partial S_1}{\partial x_\beta}$$

and, for $n \geq 2$,

$$y_{\beta,n} = \frac{\partial S_n}{\partial x_\beta} - \sum_{k=1}^{n-1} \frac{1}{k!} \sum_{\alpha_1=1}^{t_1} \cdots \sum_{\alpha_k=1}^{t_k} \frac{\partial^k y_{\beta,\alpha}}{\partial x_{i_1} \cdots \partial x_{i_k}} \frac{\partial S_{\alpha_1}}{\partial y_{i_1}} \cdots \frac{\partial S_{\alpha_k}}{\partial y_{i_k}}$$

Thus, for example,

$$y_{\beta,2} = \frac{\partial S_2}{\partial x_\beta} - \frac{\partial y_{\beta,1}}{\partial x_i} \frac{\partial S_1}{\partial y_i}$$

$$y_{\beta,3} = \frac{\partial S_3}{\partial x_\beta} - \frac{\partial y_{\beta,1}}{\partial x_i} \frac{\partial S_2}{\partial y_i} - \frac{\partial y_{\beta,2}}{\partial x_i} \frac{\partial S_1}{\partial y_i} - \frac{1}{2} \frac{\partial^2 y_{\beta,1}}{\partial x_i \, \partial x_j} \frac{\partial S_1}{\partial y_i} \frac{\partial S_1}{\partial y_j}$$

and so forth, again with $S_k = S_k(x, y)$.

It may be noted that the authors of recent papers on the Lie transformation (refs. 10 and 11) have remarked that the von Zeipel transformation is unsatisfactory because of its implicit nature. Such remarks are no longer valid in view of the present explicit algorithms for effecting the transformation of coordinates and of arbitrary functions in either direction.

REFERENCES

1. Brouwer, Dirk: Solution of the Problem of Artificial Satellite Theory Without Drag. Astronom. J., vol. 64, no. 1274, Nov. 1959, pp. 378-397.

2. Hori, Gen-ichiro: A New Approach to the Solution of the Main Problem of the Lunar Theory. Astronom. J., vol. 68, no. 1308, April 1963, pp. 125-146.

3. Brouwer, Dirk; and Clemence, Gerald M.: Methods of Celestial Mechanics. Academic Press, New York, 1961.

4. Kovalevsky, J.: Sur la Theorie du mouvement d'un Satellite a Fortes Inclinaison et Excentricite. The Theory of Orbits in the Solar System and in Stellar Systems. Academic Press, New York, 1966, pp. 326-344.

5. Szebehely, V. G.: Theory of Orbits, The Restricted Problem of Three Bodies. Academic Press, New York, 1967.

6. Smart, W. M.: Celestial Mechanics. Longmans, London, 1960.

7. Izsak, Imre G.: Tesseral Harmonics of the Geopotential and Corrections to Station Coordinates. J. Geophys. Res., vol. 69, no. 12, June 15, 1964, pp. 2621-2630.

8. Garfinkel, Boris: Tesseral Harmonic Perturbations of an Artificial Satellite. Astronom. J., vol. 70, no. 1335, Dec. 1965, pp. 784-786.

9. Garfinkel, Boris: The Disturbing Function for an Artificial Satellite. Astronom. J., vol. 70, no. 1334, Nov. 1965, pp. 699-704.

10. Hori, Gen-ichiro: Theory of General Perturbations With Unspecified Canonical Variables. Publications of the Astronomical Society of Japan, vol. 18, no. 4, 1966, pp. 287-296.

11. Deprit, André: Canonical Transformations Depending on a Small Parameter. Mathematical Note No. 574, Mathematics Research Laboratory, Boeing Scientific Research Laboratories, Sept. 1968.

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